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RANKING BY INVERSION: A NOTE ON C.L. DODGSON

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SUMMARIES

A discussion is given of three pamphlets on elections and committees by C. L. Dodgson (Lewis Carroll) written between 1873 and 1876. It is argued that Dodgson's work on cycles anticipates a stochastic model proposed by Thompson and Remage in 1964 and includes ideas that are basic to maximum likelihood estimation.

Dans cet article, nous analysons trois brochures sur les modes de scrutin et les comités écrites entre 1873 et 1876 par C. L. Dodgson (Lewis Carroll). Nous soutenons que l'étude des cycles par Dodgson anticipe le modèle stochastique proposé en 1964 par Thompson et Remage et contient des idées fondamentales pour l'estimation de la probabilité maximale.

Zur Diskussion stehen drei Broschüren über Auswahlen und Kommitees von C. L. Dodgson (Lewis Carroll), die in der Zeit von 1873 bis 1876 geschrieben wurden. Es wird argumentiert, dass Dodgsons Arbeit über Cycles ein von Thompson und Remage in 1964 vergeschlagenes Modell der Wahrscheinlichkeitsrechnung voraussieht und Ideen einschliesst, welche die Grundlage für das Maximumproblem der Wahrscheinlichkeitsrechnung bilden.

INTRODUCTION

As a mathematical logician and literary figure, C. L. Dodgson (Lewis Carroll) has a well-established reputation. However, another of his serious pursuits--his work on elections and committees--has received less attention than it deserves. In 15 years Dodgson wrote some eighteen papers, letters, articles, and pamphlets on different aspects of this topic [1]. Here we are interested in the three pamphlets written between 1873 and 1876: *A Discussion of the Various Methods of Procedure in Conducting Elections* [1873], *Suggestions as to the Best Method of Taking Votes, Where More than Two Issues Are to be Voted on* [1874], *and A Method of Taking Votes on More than Two Issues* [1876]. Duncan Black [1958] thought that Dodgson knew a great

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deal about elections generally and cyclical majorities particularly, so much so that he appended these three pamphlets to his own work on committees and elections. But it is not easy to glean the exact nature of Dodgson's contributions on cyclical majorities from Black's commentary. Aside from suggesting that Dodgson began to consider the subject in 1871, possibly earlier, that some of his procedures were actually used in carrying out the business of Christ Church, and that the third pamphlet showed him to know much more about cyclical majorities than he actually wrote, Black says little more about the nature of his contribution. Peter Fishburn [1973] has examined the proposed election procedures in the paper of 1873, and has constructed an election method based on the discussion of cyclical majorities in the paper of 1876, which Fishburn calls "Dodgson's Function." This method is based on simple majorities under inversion. As far as I am aware, this is the only analysis of Dodgson's work on cycles that has been done.

In this paper we will examine Dodgson's inversion approach and argue that it anticipates a stochastic model developed by Thompson and Remage in 1964 which uses a ranking criterion proposed by Slater in 1961.

THE METHOD OF INVERSION

One of the examples used by Dodgson, on pp. 220-230 of Black's book, has four candidates and 23 electors who have ranked the candidates in the order:

Two electors :	*a b d c*
Four electors:	*a c b d*
One electors :	*a d b c*
Six electors :	*b d a c*
Five electors:	*c b a d*
One elector :	*c b d a*
Two electors :	*d b a c*
Two electors :	*d c b a*

FIGURE 1

The majorities are cyclical and linearly ordered as *adcba*. He gives a table of these majorities as a matrix where the number of votes for a candidate appears as the numerator of the fraction in the candidate's *column*, the number against as the denominator.

TABLE 1

	a	*b*	*c*	*d*
a		16/7	8/15	11/12
b	7/16		12/11	5/18
c	15/8	11/12		13/10
d	12/11	18/5	10/13	

Dodgson discusses two possible voting solutions to the problem of cyclical majorities, one of which (plurality) would select *a*, and the other (successive elimination) would pick *c* as the winner. To show that neither solution is a good one he refers to Table 1 and considers the number of changes of votes each candidate needs to win. For *a* to win, he needs five votes in row 2, column 1 to give him a majority over *b*. To win, *b* needs one vote in (3,2); *c* needs six votes, four in (1,3) and two in (4,3), to win, while *d* needs eight votes, one in (1,4) and seven in (3,4). Dodgson claims that *b* should be the winner by this inversion approach.

DODGSON'S FUNCTION

Fishburn's analysis uses the idea of a set of n-tuples of linear orders D defined on the set of candidates X. If D_1, $D_2 \in D$, then an inversion exists when for any x, $y \in X$, we have both xD_1y and yD_2x. In the example above, if D_1 is *cbad* and D_2 is *dbac*, there are three inversions, $cb \to bc$, $bd \to db$, $ca \to ac$. Now given (X,D), let $t(x,X,D)$ be the smallest number of inversions necessary to restrict the order relation $>_i$ on X such that we obtain an order D_i for which there exists a strict simple majority choice $P(X,D_i) = \{x\}$. Dodgson's function can be written as $F(X,D) = \{x \mid x \in X \text{ and } t(x,X,D) \leq t(y,X,D) \; \forall \; y \in X\}$. In the above example $X = \{a,b,c,d\}$, and D is the set of all quadruples in Figure 1. For *b* to win a majority of 12 votes: invert $ab \to ba$ in one of the orders *abdc* and $cb \to bc$ in all of *cbad*. Then $t(b,X,D) = 6$. For *c* to win, invert $dc \to cd$ in all of *dcba* and $ac \to ca$ in all of *acbd*. Hence $t(c,X,D) = 6$. For *a* to win, 10 inversions are required; for *d*, nine are needed. So Dodgson's function selects *b* or *c*.

ESTIMATING A RANKING FROM PAIRED COMPARISONS

How did Dodgson obtain the cyclical order *adcba*? It appears that he used the table of majorities which produces the pairwise preferences $a > c$, $a > d$, $b > a$, $b > d$, $c > b$, $d > c$. Following this, he used the transitivity of the order relation and selected a maximal cycle. However, he found it difficult to reconcile cycles with the clear selection of a winner. Dodgson's goal was to find the "best" candidate. In pursuit of that goal he became involved with the problem of preferential order. He was also deeply concerned with cyclical majorities and their effect on choosing the best candidate. In fact, he flatly states that when majorities remain cyclical after several remedial steps, there should be no election. In the third pamphlet the following measures are proposed:

1. If no candidate wins by majority vote over all other candidates taken separately, then further debate should take place.

2. If there is still no resolution, then a cycle of candidates should be formed such that each member of the cycle separately beats each candidate not in the cycle.

3. When the cadidates have been placed in a single cycle, inversion should be used to indicate to each elector the number of votes each candidate needs to win by a majority vote. Changes in the votes are then allowed [Black 1958, 224-225, "Proposed Rules for Conducting an Election"].

Cyclical majorities generally result from paired comparison methods and rarely from weighted ranking systems like the "method of marks" which Dodgson discusses in the first pamphlet [2]. Dodgson's work in the second and third pamphlets is supportive of this interpretation. He writes:

> ... *some one issue may be discovered, which is preferred by a majority to every other taken separately. For this purpose, any two may be put up to begin with, then the winning issue along with some other and so on. But no issue can be considered as the absolute winner, unless it has been put up along* every *other* [from the section, "Failing a settlement by this method, the issues to be voted on two at a time," Black 1958, 223-224]. It seems to me that this [Ordinance] may be compiled with by either of two modes of election:

In case (α) *If a candidate be declared elected who, when all are voted on at once, has an absolute majority of votes.*
In case (β) *If a candidate be declared elected who, when paired with every other separately, is preferred by the majority of those voting* [from the third pamphlet, the section "The legal conditions," Black 1958, 226].

Dodgson was really grappling with the question: When are sets of paired comparisons consistent with a unique ranking of the candidates? He was undoubtedly aware of the absence of any generally accepted definition of a "true" ranking. What he did, as I shall now argue, was to select the winner as the first element in a unique ranking based on a probability model which he could only have known intuitively [3].

Binary comparisons involve comparing in pairs a set of $X = \{x_1, x_2, \ldots, x_m\}$ items independently by n_{ij} judges in n_{ij} independent trials. The judges establish one of the preference relations $x_i > x_j$ or $x_j > x_i$, for all pairs of elements of X. The set of preferences can be considered as a sample of the population parameters,

$$\pi_{ij} = P(x_i > x_j),\ \pi_{ij} + \pi_{ji} = 1.$$

If $\pi_{ij} > 1/2$, x_i is said to be *stochastically preferred* to x_j in the population. The preference order $\{x_i, x_2, \ldots, x_m\}$ is called *weakly stochastic* if $\pi_{ij} \geq 1/2$, $i < j$. Thompson and Remage determined the maximum-liklihood weak stochastic order by maximizing over the Π_{ij} the function

$$\sum_{i > j} \binom{n_{ij}}{a_{ij}} \prod\nolimits_{ij}^{a_{ij}} \quad \prod\nolimits_{ji}^{a_{ji}}$$

with the constraint that there exists no cycle $i_1, i_2, \ldots, i_k, i_1$ such that $\Pi_{i_1 i_2} > 1/2$, $\Pi_{i_2 i_3} > 1/2, \ldots,$ $\Pi_{i_k i_1} > 1/2$. The a_{ij} express the number of times $x_i > x_j$, $a_{ii} = 0$, $a_{ij} + a_{ji} = n_{ij}$. The authors show that this maximum can be obtained in different ways, so that the "best" order is not necessarily unique. However, when $n_{ij} = 1$ for all $i \neq j$, then the best order is Slater's nearest adjoining order.

In order to demonstrate the difficulty of selecting a winner, Dodgson subjects various cases involving cyclical majorities to his inversion criterion. When considered as a procedure to rank all the competing candidates, it is a maximum-likelihood weak stochastic ranking process. In this sense it is a best ranking. This is illustrated by one of Dodgson's examples [Black 1958, 228-229]. There are 15 judges and four candidates.

TABLE 2

$$A = \begin{bmatrix} 0 & 8 & 6 & 4 \\ 7 & 0 & 9 & 4 \\ 9 & 6 & 0 & 8 \\ 11 & 11 & 7 & 0 \end{bmatrix}$$

TABLE 3

$$P = \begin{bmatrix} 0 & 8/15 & 6/15 & 4/15 \\ 7/15 & 0 & 9/15 & 4/15 \\ 9/15 & 6/15 & 0 & 8/15 \\ 11/15 & 11/15 & 7/15 & 0 \end{bmatrix}$$

TABLE 4. Table of Majorities

	a_1	a_2	a_3	a_4
a_1		7/8	9/6	11/4
a_2	8/7		6/9	11/4
a_3	6/9	9/6		7/8
a_4	4/11	4/11	8/7	

Assume that the candidates are compared pairwise, once by each judge, and that each judge's preference list gives rise to a set of paired comparisons; for example, the ordered list a_1, a_3, a_4, a_2 produces the set $\{(a_1,a_3),(a_1,a_4), (a_1,a_2), (a_3,a_4), (a_3,a_2), (a_4,a_2)\}$. The 15 sets are then combined into a single preference matrix A from which one ranking is computed. The probability matrix P associated with $A = (a_{ij})$ is then the transpose, suitably adjusted, of the table of majorities (refer to Tables 2-4).

Candidate 1 can win with 6 inversions (see Table 5); candidate 2 with 5 inversions (Table 6); candidate 3 requires 2 inversions (Table 7); candidate 4 needs only 1 inversion (Table 8). For a_i to win he must beat every other candidate. In this example a_4 wins because in doing so the least change occurs in producing the maximal four path, $a_4 > a_2 > a_3 > a_1$. In fact if inversion is applied successively to the remaining candidates, this path is the result. It is the unique maximum likelihood weak stochastic rank order.

TABLE 5

$$A_1 = \begin{bmatrix} 0 & 8 & 8 & 8 \\ 7 & 0 & 9 & 4 \\ 7 & 6 & 0 & 8 \\ 7 & 11 & 7 & 0 \end{bmatrix}$$

TABLE 6

$$A_2 = \begin{bmatrix} 0 & 7 & 6 & 4 \\ 8 & 0 & 9 & 8 \\ 9 & 6 & 0 & 8 \\ 11 & 7 & 7 & 0 \end{bmatrix}$$

TABLE 7

$$A_3 = \begin{bmatrix} 0 & 8 & 6 & 4 \\ 7 & 0 & 7 & 4 \\ 9 & 8 & 0 & 8 \\ 11 & 11 & 7 & 0 \end{bmatrix}$$

TABLE 8

$$A_4 = \begin{bmatrix} 0 & 8 & 6 & 4 \\ 7 & 0 & 9 & 4 \\ 9 & 6 & 0 & 7 \\ 11 & 11 & 8 & 0 \end{bmatrix}$$

In this paper we have examined the approach that Dodgson used to overcome the problem of cyclical majorities. He had an appreciation of the difficulties that was well in advance of his time. Dodgson worked with a clever scheme that contained the nucleus of a solution to the problem of estimating the maximum-likelihood weak stochastic ranking from a sample of paired comparisons. But he did not contribute directly to its development.

NOTES

1. In addition to the three pamphlets cited, there are: "The Cyclostyled Sheet" 1877, *The Principles of Parliamentary Representation* 1884, the second edition of this book 1885, a supplement to it 1885, a postscript to the supplement 1885, Lawn Tennis Tournaments 1883, nine letters and articles which appeared between 1881 and 1885 in *The St. James Gazette*.

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2. One kind of weighted ranking system is given $n > 3$ candidates, the one ranked first is scored n, the next $n - 1, \ldots,$ the last 0. The candidate with the highest score wins. Dodgson's "method of marks" is somewhat different. It can be found in [Black 1958, 218].

3. Dodgson was familiar with Todhunter's history of probability (see References), but there is nothing in it to suggest any connection with his work on cyclical majorities.

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